

Chapter 2

Rods and Trusses

Abstract This chapter treats the rod or bar member. The three basic equations of continuum mechanics, i.e. the kinematics relationship, the constitutive law and the equilibrium equation, are summarized as well as the describing partial differential equation. A general solution for constant material and geometrical properties is presented. Furthermore, typical boundary conditions and the internal reactions are briefly mentioned. The finite element formulation is focused on rod elements with two nodes under the assumption of constant material and geometrical properties. The post-computation, which is based on the nodal results, is treated in detail. The chapter concludes with the spatial arrangement of rod elements in a plane to form a plane truss structure.

2.1 Fundamentals and Analytical Treatment

A rod is defined as a prismatic body whose axial dimension is much larger than its transverse dimensions [2, 10, 14, 34, 36]. This structural member is only loaded in the direction of the main body axes, see Fig. 2.1a. As a result of this loading, the deformation occurs only along its main axis.

Derivations are restricted many times to the following simplifications:

- only applying to straight rods,
- displacements are (infinitesimally) small,
- strains are (infinitesimally) small, and
- material is linear-elastic.

The three basic equations of continuum mechanics, i.e. the kinematics relationship, the constitutive law and the equilibrium equation, as well as their combination to the describing partial differential equation (PDE) are summarized in Table 2.1.

Under the assumption of constant material ($E = \text{const.}$) and geometric ($A = \text{const.}$) properties, the differential equation in Table 2.1 can be easily integrated twice for constant distributed load ($p_x = p_0 = \text{const.}$) to obtain the general solution of the problem [25]:

Fig. 2.1 Schematic representation of a continuum rod

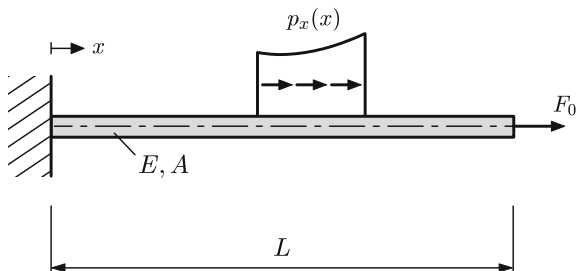


Table 2.1 Different formulations of the basic equations for a rod (x -axis along the principal rod axis), with $\mathcal{L}_1(\dots) = \frac{d(\dots)}{dx}$

Specific formulation	General formulation
Kinematics	
$\varepsilon_x(x) = \frac{du_x(x)}{dx}$	$\varepsilon_x(x) = \mathcal{L}_1(u_x(x))$
Constitution	
$\sigma_x(x) = E\varepsilon_x(x)$	$\sigma_x(x) = C\varepsilon_x(x)$
Equilibrium	
$\frac{d\sigma_x(x)}{dx} + \frac{p_x(x)}{A} = 0$	$\mathcal{L}_1^T(\sigma_x(x)) + b = 0$
PDE	
$\frac{d}{dx}\left(E(x)A(x)\frac{du_x}{dx}\right) + p_x(x) = 0$	$\mathcal{L}_1^T(EA\mathcal{L}_1(u_x(x))) + p_x = 0$

$$u_x(x) = \frac{1}{EA} \left(-\frac{1}{2} p_0 x^2 + c_1 x + c_2 \right), \quad (2.1)$$

where the two constants of integration c_i ($i = 1, 2$) must be determined based on the boundary conditions (see Table 2.2). The following equation for the internal normal force N_x was obtained based on one-time integration of the PDE and might be useful to determine some of the constants of integration:

$$N_x(x) = EA \frac{du_x(x)}{dx} = -p_0 x + c_1. \quad (2.2)$$

The internal reactions in a rod become visible if one cuts—at an arbitrary location x —the member in two parts. As a result, two opposite oriented normal forces N_x can be indicated. Summing up the internal reactions from both parts must result in zero. Their positive direction is connected with the direction of the outward surface normal vector and the orientation of the positive x -axis, see Fig. 2.2.

Once the internal normal force N_x is known, the normal stress σ_x can be calculated:

Table 2.2 Different boundary conditions and corresponding reactions for a continuum rod

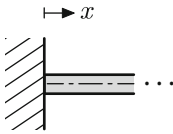
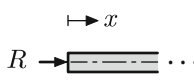
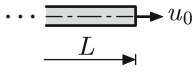
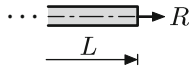
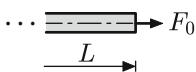
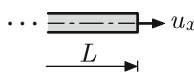
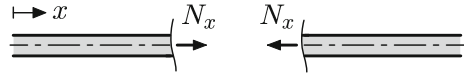
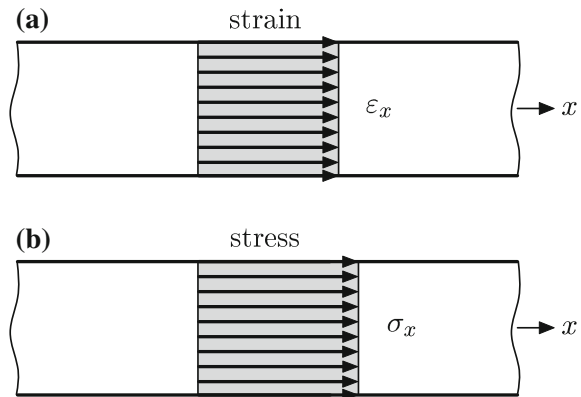
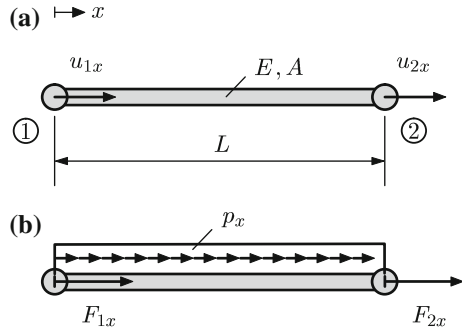
Case	Boundary Condition	Reaction
	$u_x(x=0) = 0$	
	$u_x(x=L) = u_0$	
	$EA \frac{du_x(L)}{dx} = N_x(L) = F_0$	

Fig. 2.2 Internal reactions for a continuum rod**Fig. 2.3** Axially loaded rod: **a** strain and **b** stress distribution

$$\sigma_x(x) = \frac{N_x(x)}{A}. \quad (2.3)$$

Application of Hooke's law (see Table 2.1) allows to calculate the strains ε_x . Typical idealized distributions of stress and strain in a rod element are shown in Fig. 2.3. It can be seen that both distributions are constant over the cross section.

Fig. 2.4 Definition of the one-dimensional linear rod element: **a** deformations; **b** external loads. The nodes are symbolized by the two circles at the ends (○)



2.2 Rod Elements

2.2.1 Revision of Theory

Let us consider in the following a rod element which is composed of two nodes as schematically shown in Fig. 2.4. Each node has only one degree of freedom, i.e. a displacement u_x in the direction of the x -axis (i.e., the direction of the principal axis, see Fig. 2.4a) and each node can be only loaded by single forces acting in x -direction (cf. Fig. 2.4b). In the case of distributed loads $p_x(x)$, a transformation to equivalent nodal loads is required.

Different methods can be found in the literature to derive the principal finite element equation (see [8, 24]). All these methods result in the same formulation, which is given in the following for constant material and geometrical properties:

$$\frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_{1x} \\ u_{2x} \end{bmatrix} = \begin{bmatrix} F_{1x} \\ F_{2x} \end{bmatrix} + \int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} p_x(x) dx, \quad (2.4)$$

or in abbreviated form

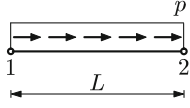
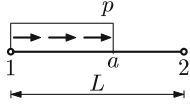
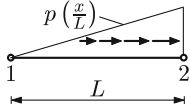
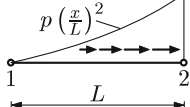
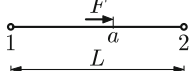
$$\mathbf{K}^e \mathbf{u}_p^e = \mathbf{f}^e, \quad (2.5)$$

where $\mathbf{K}^e$ is the elemental stiffness matrix, $\mathbf{u}_p^e$ is the elemental column matrix of unknowns and $\mathbf{f}^e$ is the elemental column matrix of loads. The interpolation functions in Eq. (2.4) are given by $N_1(x) = 1 - \frac{x}{L}$ and $N_2(x) = \frac{x}{L}$ and Table 2.3 summarizes the equivalent nodal loads for some simple shapes of distributed loads.

Several single finite elements can be combined to form a finite element mesh and the assembly of the elemental equations results in the global system of equations, i.e.

$$\mathbf{K} \mathbf{u}_p = \mathbf{f}, \quad (2.6)$$

Table 2.3 Equivalent nodal loads for a linear rod element (x -axis: right facing)

Loading	Equivalent Axial Force
	$F_{1x} = \frac{pL}{2}$ $F_{2x} = \frac{pL}{2}$
	$F_{1x} = -\frac{pa^2}{2L} + pa$ $F_{2x} = \frac{pa^2}{2L}$
	$F_{1x} = \frac{pL}{6}$ $F_{2x} = \frac{pL}{3}$
	$F_{1x} = \frac{pL}{12}$ $F_{2x} = \frac{pL}{4}$
	$F_{1x} = \frac{F(L-a)}{L}$ $F_{2x} = \frac{Fa}{L}$

where $\mathbf{K}$ is the global stiffness matrix, $\mathbf{u}_p$ is the global column matrix of unknowns and $\mathbf{f}$ is the global column matrix of loads. The global system of equations in the form of Eq. (2.6) cannot be solved without the consideration of the support conditions (this results in the reduced system of equations). A few methods to consider different types of boundary conditions are summarized in the following:

- Homogenous Dirichlet boundary condition $u_x = 0$

A homogenous Dirichlet¹ boundary condition at node n ($u_{nX} = 0$) can be considered in the non-reduced system of equations by eliminating the n th row and n th column of the system, see Eq. (2.7).

¹Alternatively known as 1st kind, essential, geometric or kinematic boundary condition.

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & n & & \\
 \left[\begin{array}{ccccc}
 & & \text{shaded} & & \\
 & & \times & & \\
 & & \times & & \\
 \times & \times & \times & \times & \times \\
 & & \times & & \\
 & & \times & & \\
 & & \times & &
 \end{array} \right] & \begin{bmatrix} \\ \\ \\ u_{nX} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \text{shaded} \\ \\ \end{bmatrix} \cdot
 \end{array}
 \end{array} \quad (2.7)$$

- Non-homogeneous Dirichlet boundary condition $u_x \neq 0$

First possibility: A non-homogeneous Dirichlet boundary condition ($u_{nX} = u_0 \neq 0$) at node n can be introduced in the system of equations by modifying the n th row in such a way that at the position of the n th column a '1' is obtained while all other entries of the n th row are set to zero. On the right-hand side, the given value u_0 is introduced at the n th position of the column matrix of the external loads as follows:

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & n & & \\
 \left[\begin{array}{ccccc}
 & & & & \\
 & & & & \\
 0 & 0 & 1 & 0 & 0 \\
 & & & & \\
 & & & &
 \end{array} \right] & \begin{bmatrix} \\ \\ u_{nX} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ u_0 \\ \\ \end{bmatrix} \cdot
 \end{array}
 \end{array} \quad (2.8)$$

Second possibility: If the boundary condition is specified at node n , the n th column of the stiffness matrix is multiplied by the given value u_0 . Now we bring the n th column of the stiffness matrix to the right-hand side of the system and delete the n th row of the system of equations. These steps can be identified in the following equations:

$$\begin{array}{c}
 \begin{array}{ccccc}
 & n-1 & n & n+1 & \\
 \left[\begin{array}{ccccc}
 & & (\dots)u_0 & & \\
 & & (\dots)u_0 & & \\
 & & (\dots)u_0 & & \\
 & & (\dots)u_0 & & \\
 & & (\dots)u_0 & &
 \end{array} \right] & \begin{bmatrix} \\ \\ u_{n-1X} \\ u_{nX} \\ u_{n+1X} \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}
 \end{array} \quad (2.9)$$

$$\Rightarrow \begin{array}{ccccc}
 & n-1 & n+1 & & \\
 \left[\begin{array}{ccccc}
 & & & & \\
 & & & & \\
 & & & & \\
 & & & & \\
 & & & &
 \end{array} \right] & \begin{bmatrix} \\ \\ u_{n-1X} \\ u_{nX} \\ u_{n+1X} \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} - \begin{bmatrix} (\dots)u_0 \\ (\dots)u_0 \\ (\dots)u_0 \\ (\dots)u_0 \\ (\dots)u_0 \end{bmatrix}
 \end{array} \quad (2.10)$$

$$\Rightarrow \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} u_{n-1X} \\ u_{n+1X} \end{bmatrix} = \begin{bmatrix} -(\dots)u_0 \\ -(\dots)u_0 \\ -(\dots)u_0 \\ -(\dots)u_0 \end{bmatrix}. \quad (2.11)$$

Third possibility: Replace in the column matrix of unknowns the variable of the nodal value u_{nX} with the given value u_0 and introduce in the column matrix of the external loads at the n th position the corresponding reaction force R_{nX} . Split the column matrix of the external loads into a component with the given external loads and a component which contains the unknown reaction force R_{nX} . Now we bring the n th column of the stiffness matrix to the right-hand side of the system and the component of the load matrix with R_{nX} to the left-hand side:

$$\begin{bmatrix} & & n & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} u_0 \end{bmatrix} = \begin{bmatrix} R_{n,X} \end{bmatrix}. \quad (2.12)$$

- Neumann boundary condition $F_x = F_0$

A Neumann² boundary condition at node n ($F_{nX} = F_0$) can be considered on the right-hand side, i.e. in the column matrix of the external loads.

Once the nodal displacements (u_{1X}, u_{2X}) are known, further quantities and their distributions can be calculated within an element (so-called post-processing), see Table 2.4.

Let us summarize here the recommended steps for a linear finite element solution ('hand calculation'):

- ① Sketch the free-body diagram of the problem, including a global coordinate system.
- ② Subdivide the geometry into finite elements. Indicate the node and element numbers (the user may choose any numbering order), local coordinate systems, and equivalent nodal loads.
- ③ Write separately all elemental stiffness matrices expressed in the global coordinate system. Indicate for each element the nodal unknowns (degrees of freedom) on the right-hand side and over the matrix. In this step, the DOFs must be chosen according to the global coordinate system—conventionally, in the positive direction.

²Alternatively known as 2nd kind, natural or static boundary condition.

Table 2.4 Post-processing of nodal values for a linear rod element (defined by element length L , cross-sectional area A , and Young's modulus E). The distributions are given as being dependent on the nodal values as a function of the physical coordinate $0 \leq x \leq L$ and the natural coordinate $-1 \leq \xi \leq 1$

Axial displacement (elongation) u_x	
$u_x^e(x) = \left[1 - \frac{x}{L}\right] u_{1x} + \left[\frac{x}{L}\right] u_{2x}$	
$u_x^e(\xi) = \left[\frac{1}{2}(1 - \xi)\right] u_{1x} + \left[\frac{1}{2}(1 + \xi)\right] u_{2x}$	
Axial Strain $\varepsilon_x = \frac{du_x}{dx} = \frac{d\xi}{dx} \frac{du_x}{d\xi}$	
$\varepsilon_x^e(x) = \frac{1}{L} (u_{2x} - u_{1x})$	
$\varepsilon_x^e(\xi) = \frac{1}{L} (u_{2x} - u_{1x})$	
Axial Stress $\sigma_x = E\varepsilon_x = E \frac{du_x}{dx} = E \frac{d\xi}{dx} \frac{du_x}{d\xi}$	
$\sigma_x^e(x) = \frac{E}{L} (u_{2x} - u_{1x})$	
$\sigma_x^e(\xi) = \frac{E}{L} (u_{2x} - u_{1x})$	
Normal Force $N_x = EA\varepsilon_x = EA \frac{du_x}{dx} = EA \frac{d\xi}{dx} \frac{du_x}{d\xi}$	
$N_x^e(x) = \frac{EA}{L} (u_{2x} - u_{1x})$	
$N_x^e(\xi) = \frac{EA}{L} (u_{2x} - u_{1x})$	

- ④ Determine the dimensions of the global stiffness matrix and sketch the structure of this matrix with global unknowns on the right-hand side and over the matrix. The dimensions of the matrix are equal to the total number of degrees of freedom which can be determined by multiplying the number of nodes by the number of degrees of freedom per node. After assembling, the validity of the assembled stiffness matrix can be tested by the following check list:
 - $\mathbf{K}$ is symmetrical,
 - $\mathbf{K}$ has only positive components on the main diagonal, and
 - the coupled DOFs have non-zero values as their corresponding components.
- ⑤ Insert step-by-step the values of the elemental stiffness matrices into the global stiffness matrix. This process is called ‘assembling’ the global stiffness matrix.
- ⑥ Add the column matrix of unknowns and external loads to complete the global system of equations.
- ⑦ Introduce the boundary conditions to obtain the reduced system of equations.
- ⑧ Solve the reduced system of equations to obtain the unknown nodal deformations.
- ⑨ Post-computation or post-processing: determination of reaction forces, stresses and strains.

Fig. 2.5 Rod structure with a point load

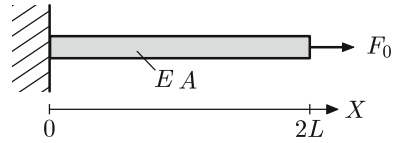


Fig. 2.6 Free-body diagram of the rod structure with a point load



- ⑩ Check the global equilibrium between the external loads and the support reactions (optional step for checking the results).

It should be noted that some steps may be combined or omitted depending on the problem and the experience of the finite element user. The above steps can be seen as an initial structured guide to master the solution of finite element problems.

2.2.2 Worked Examples

2.1 Example: Rod Structure with a Point Load

Given is a rod structure as shown in Fig. 2.5. The structure has a uniform cross-sectional area A and Young's modulus E . The structure is fixed at its left-hand end and loaded by a single force F_0 .

Model the rod structure with two linear finite elements of equal length L and determine:

- the displacements at the nodes,
- the reaction force at the left-hand support,
- the strain, stress, and normal force in the element and
- check the global force equilibrium.

2.1 Solution

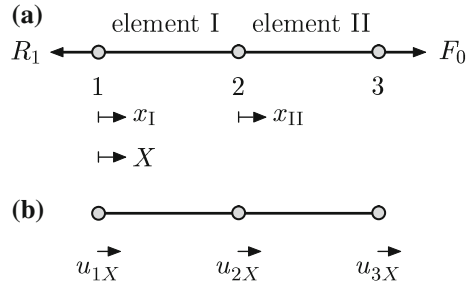
The solution will follow the recommended 10 steps outlined on page 23.

- ① Sketch the free-body diagram of the problem, including a global coordinate system.

Remove the support at the left-hand end and introduce the corresponding reaction force, see Fig. 2.6. Note that the direction of the reaction force can be arbitrarily chosen. The sign of the result will confirm ($R_1 > 0$) or reject ($R_1 < 0$) the assumed direction.

The rod structure with a total length of $2L$ is divided in the middle into two elements, see Fig. 2.7.

Fig. 2.7 **a** Free-body diagram of the discretized structure with point loads. **b** Nodal unknowns



③ Write separately all elemental stiffness matrices expressed in the global coordinate system. Indicate the nodal unknowns on the right-hand sides and over the matrices.

$$\mathbf{K}_I^e = \frac{EA}{L} \begin{bmatrix} u_{1X} & u_{2X} \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} u_{1X} \\ u_{2X} \end{matrix}, \quad (2.13)$$

$$\mathbf{K}_{II}^e = \frac{EA}{L} \begin{bmatrix} u_{2X} & u_{3X} \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} u_{2X} \\ u_{3X} \end{matrix}. \quad (2.14)$$

④ Determine the dimension of the global stiffness matrix and sketch the structure of this matrix with global unknowns on the right-hand side and over the matrix.

The finite element structure is composed of 3 nodes, each having one degree of freedom (i.e., the axial displacement). Thus, the dimension of the global stiffness matrix is $(3 \times 1) \times (3 \times 1) = (3 \times 3)$:

$$\mathbf{K} = \begin{bmatrix} u_{1X} & u_{2X} & u_{3X} \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{bmatrix} \begin{matrix} u_{1X} \\ u_{2X} \\ u_{3X} \end{matrix}. \quad (2.15)$$

⑤ Insert the values of the elemental stiffness matrices step-by-step into the global stiffness matrix:

$$\mathbf{K} = \frac{EA}{L} \begin{bmatrix} u_{1X} & u_{2X} & u_{3X} \\ 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{matrix} u_{1X} \\ u_{2X} \\ u_{3X} \end{matrix}. \quad (2.16)$$

⑥ Add the column matrix of unknowns and external loads to complete the global system of equations:

$$\frac{EA}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_{1X} \\ u_{2X} \\ u_{3X} \end{bmatrix} = \begin{bmatrix} -R_1 \\ 0 \\ F_0 \end{bmatrix}. \quad (2.17)$$

⑦ Introduce the boundary conditions to obtain the reduced system of equations. There is no displacement possible at the left-hand end of the structure (i.e., $u_{1X} = 0$ at node 1). Thus, cancel the first row and first column from the linear system to obtain:

$$\frac{EA}{L} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{3X} \end{bmatrix} = \begin{bmatrix} 0 \\ F_0 \end{bmatrix}. \quad (2.18)$$

⑧ Solve the reduced system of equations to obtain the unknown nodal deformations.

The solution can be obtained based on the matrix approach $\mathbf{u} = \mathbf{K}^{-1} \mathbf{f}$:

$$\begin{bmatrix} u_{2X} \\ u_{3X} \end{bmatrix} = \frac{L}{EA} \times \frac{1}{2-1} \begin{bmatrix} +1 & +1 \\ +1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ F_0 \end{bmatrix} = \frac{F_0 L}{EA} \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \quad (2.19)$$

⑨ Post-computation: determination of reaction forces, stresses and strains.

Take into account the non-reduced system of equations as given in step ⑥ under the consideration of the known nodal displacements. The first equation of this system reads:

$$\frac{EA}{L} (-u_{2X}) = -R_1, \quad (2.20)$$

or finally for the reaction force:

$$R_1 = F_0. \quad (2.21)$$

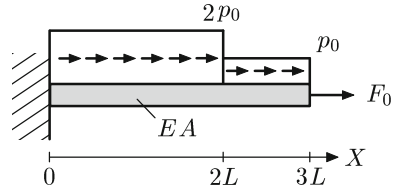
The obtained positive value confirms the assumption of the selected initial direction for the reaction force.

The equations for the elemental strains, stresses, and normal forces can be extracted from Table 2.4:

$$\varepsilon_I^e = \frac{1}{L} (u_{2X} - u_{1X}) = \frac{F_0}{EA}, \quad (2.22)$$

$$\varepsilon_{II}^e = \frac{1}{L} (u_{3X} - u_{2X}) = \frac{F_0}{EA}, \quad (2.23)$$

Fig. 2.8 Rod structure with changing distributed load



$$\sigma_I^e = \frac{E}{L} (u_{2X} - u_{1X}) = \frac{F_0}{A}, \quad (2.24)$$

$$\sigma_{II}^e = \frac{E}{L} (u_{3X} - u_{2X}) = \frac{F_0}{A}, \quad (2.25)$$

$$N_I^e = \frac{EA}{L} (u_{2X} - u_{1X}) = F_0, \quad (2.26)$$

$$N_{II}^e = \frac{EA}{L} (u_{3X} - u_{2X}) = F_0. \quad (2.27)$$

⑩ Check the global equilibrium between the external loads and the support reactions:

$$\sum_i F_{iX} = 0 \Leftrightarrow \underbrace{-R_1}_{\text{reaction force}} + \underbrace{F_0}_{\text{external load}} = 0 \quad \checkmark \quad (2.28)$$

2.2 Example: Rod Structure with Changing Distributed Load

Given is a rod structure as shown in Fig. 2.8. The structure has a uniform cross-sectional area A and Young's modulus E . The structure is fixed at its left-hand end and loaded by a single force F_0 at $X = 3L$ as well as

- (a) a uniform distributed load $2p_0$ in the range $0 \leq X \leq 2L$, and
- (b) a uniform distributed load p_0 in the range $2L \leq X \leq 3L$.

Model the rod structure with two linear finite elements and determine

- the displacements at the nodes,
- the reaction force at the left-hand support,
- the strain, stress, and normal force in each element and
- check the global force equilibrium.

2.2 Solution

The solution will follow the recommended 10 steps outlined on page 23.

① Sketch the free-body diagram of the problem, including a global coordinate system.

Fig. 2.9 Free-body diagram of the rod structure with changing distributed load

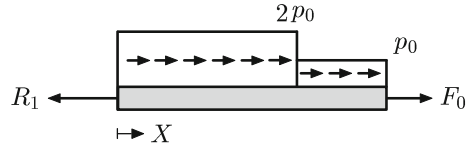
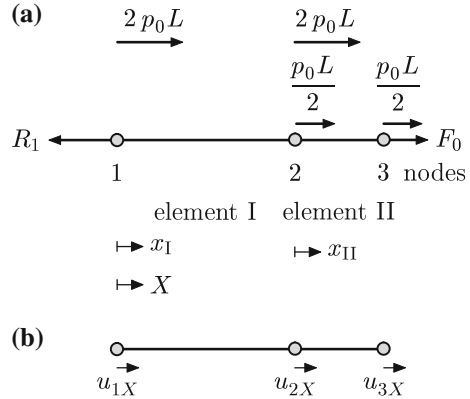


Fig. 2.10 a Free-body diagram of the discretized structure with equivalent nodal loads and **b** Nodal unknowns



Remove the support at the left-hand end and introduce the corresponding reaction force, see Fig. 2.9. Note that the direction of the reaction force can be arbitrarily chosen. The sign of the result will confirm ($R_1 > 0$) or reject ($R_1 < 0$) the assumed direction. Looking from a different angle at the problem, we can say that nodes are introduced at the locations of the single forces (R_1 and F_0) and the discontinuity of the distributed load.

② Subdivide the geometry into finite elements. Indicate the node and element numbers, local coordinate systems, and equivalent nodal loads.

The rod structure with a total length of $3L$ is divided at the discontinuity of the distributed load (i.e., at $X = 2L$) into two elements and the corresponding equivalent nodal loads are calculated from Table 2.3, see Fig. 2.10. The two force contributions of magnitude $2p_0L$ result from the distributed load $2p_0$ and the two force contributions of magnitude $\frac{p_0L}{2}$ result from the distributed load p_0 .

③ Write separately all elemental stiffness matrices expressed in the global coordinate system. Indicate the nodal unknowns on the right-hand sides and over the matrices:

$$\mathbf{K}_I^e = \frac{EA}{2L} \begin{bmatrix} & u_{1X} & u_{2X} \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} u_{1X} \\ u_{2X} \end{matrix} = \frac{EA}{L} \begin{bmatrix} & u_{1X} & u_{2X} \\ \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{matrix} u_{1X} \\ u_{2X} \end{matrix}, \quad (2.29)$$

$$\mathbf{K}_{II}^e = \frac{EA}{L} \begin{bmatrix} u_{2X} & u_{3X} \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{3X} \end{bmatrix}. \quad (2.30)$$

④ Determine the dimension of the global stiffness matrix and sketch the structure of this matrix with global unknowns on the right-hand side and over the matrix.

The finite element structure is composed of 3 nodes, each having one degree of freedom (i.e., the axial displacement). Thus, the dimension of the global stiffness matrix is $(3 \times 1) \times (3 \times 1) = (3 \times 3)$:

$$\mathbf{K} = \begin{bmatrix} u_{1X} & u_{2X} & u_{3X} \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{bmatrix} \begin{bmatrix} u_{1X} \\ u_{2X} \\ u_{3X} \end{bmatrix}. \quad (2.31)$$

⑤ Insert the values of the elemental stiffness matrices step-by-step into the global stiffness matrix:

$$\mathbf{K} = \frac{EA}{L} \begin{bmatrix} u_{1X} & u_{2X} & u_{3X} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} + 1 & -1 \\ 0 & -1 & +1 \end{bmatrix} \begin{bmatrix} u_{1X} \\ u_{2X} \\ u_{3X} \end{bmatrix}. \quad (2.32)$$

⑥ Add the column matrix of unknowns and external loads to complete the global system of equations:

$$\frac{EA}{L} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} + 1 & -1 \\ 0 & -1 & +1 \end{bmatrix} \begin{bmatrix} u_{1X} \\ u_{2X} \\ u_{3X} \end{bmatrix} = \begin{bmatrix} -R_1 + 2p_0L \\ \frac{5}{2}p_0L \\ F_0 + \frac{p_0L}{2} \end{bmatrix}. \quad (2.33)$$

⑦ Introduce the boundary conditions to obtain the reduced system of equations.

There is no displacement possible at the left-hand end of the structure (i.e., $u_{1X} = 0$ at node 1). Thus, cancel the first row and first column from the linear system to obtain:

$$\frac{EA}{L} \begin{bmatrix} \frac{3}{2} & -1 \\ -1 & +1 \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{3X} \end{bmatrix} = \begin{bmatrix} \frac{5}{2}p_0L \\ F_0 + \frac{p_0L}{2} \end{bmatrix}. \quad (2.34)$$

⑧ Solve the reduced system of equations to obtain the unknown nodal deformations.

The solution can be obtained based on the matrix approach $\mathbf{u} = \mathbf{K}^{-1} \mathbf{f}$:

$$\begin{bmatrix} u_{2X} \\ u_{3X} \end{bmatrix} = \frac{L}{EA} \times \frac{1}{\frac{3}{2} - 1} \begin{bmatrix} +1 & +1 \\ +1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \frac{5}{2} p_0 L \\ F_0 + \frac{p_0 L}{2} \end{bmatrix} = \frac{2L}{EA} \begin{bmatrix} 3p_0 L + F_0 \\ \frac{13}{4} p_0 L + \frac{3}{2} F_0 \end{bmatrix}. \quad (2.35)$$

⑨ Post-computation: determination of reaction forces, stresses and strains.

Take into account the non-reduced system of equations as given in step ⑥ under the consideration of the known nodal displacements. The first equation of this system reads:

$$\frac{EA}{L} \left(\frac{1}{2} u_{1X} - \frac{1}{2} u_{2X} + 0 \right) = -R_1 + 2p_0 L, \quad (2.36)$$

or finally for the reaction force:

$$R_1 = 5p_0 L + F_0. \quad (2.37)$$

The equations for the elemental strains, stresses, and normal forces can be extracted from Table 2.4:

$$\varepsilon_I^e = \frac{1}{L_I} (u_{2X} - u_{1X}) = \frac{1}{EA} (3p_0 L + F_0), \quad (2.38)$$

$$\varepsilon_{II}^e = \frac{1}{L_{II}} (u_{3X} - u_{2X}) = \frac{1}{EA} \left(\frac{1}{2} p_0 L + F_0 \right), \quad (2.39)$$

$$\sigma_I^e = \frac{E}{L_I} (u_{2X} - u_{1X}) = \frac{1}{A} (3p_0 L + F_0), \quad (2.40)$$

$$\sigma_{II}^e = \frac{E}{L_{II}} (u_{3X} - u_{2X}) = \frac{1}{A} \left(\frac{1}{2} p_0 L + F_0 \right), \quad (2.41)$$

$$N_I^e = \frac{EA}{L_I} (u_{2X} - u_{1X}) = 3p_0 L + F_0, \quad (2.42)$$

$$N_{II}^e = \frac{EA}{L_{II}} (u_{3X} - u_{2X}) = \frac{1}{2} p_0 L + F_0. \quad (2.43)$$

⑩ Check the global equilibrium between the external loads and the support reactions:

$$\sum_i F_{iX} = 0 \Leftrightarrow \underbrace{-(5p_0 L + F_0)}_{\text{reaction force}} + \underbrace{F_0 + 4p_0 L + p_0 L}_{\text{external loads}} = 0 \quad \checkmark \quad (2.44)$$

Fig. 2.11 Rod structure with displacement and force boundary conditions

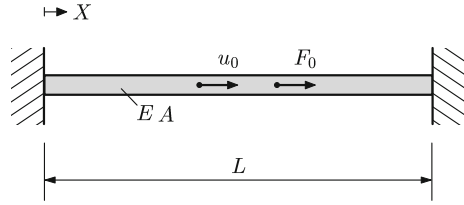
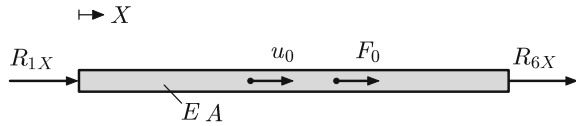


Fig. 2.12 Free-body diagram of the rod structure with displacement and force boundary condition



2.3 Example: Rod Structure with Displacement and Force Boundary Conditions

Given is a rod structure as shown in Fig. 2.11. The structure has a uniform cross-sectional area A and Young's modulus E . The structure is fixed at both ends and loaded by a single force F_0 at $X = \frac{3}{5}L$ as well as displacement u_0 at $X = \frac{2}{5}L$.

Model the rod structure with five linear finite elements of equal length and determine

- the displacements at the nodes,
- the reaction forces at the supports,
- the strain, stress, and normal force in each element and
- check the global force equilibrium.
- Assume now that only u_0 is given. Adjust the value of F_0 in such a way that element III is in a stress-free state.

2.3 Solution

The solution will follow the recommended 10 steps outlined on page 23.

- ① Sketch the free-body diagram of the problem, including a global coordinate system.

Remove the supports at both ends and introduce the corresponding reaction forces, see Fig. 2.12.

- ② Subdivide the geometry into finite elements. Indicate the node and element numbers, local coordinate systems, and equivalent nodal loads, see Fig. 2.13.

- ③ Write separately all elemental stiffness matrices expressed in the global coordinate system. Indicate the nodal unknowns on the right-hand sides and over the matrices.

$$\mathbf{K}_I^e = \frac{EA}{\frac{L}{5}} \begin{bmatrix} u_{1X} & u_{2X} \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} u_{1X} \\ u_{2X} \end{matrix}, \quad \mathbf{K}_{II}^e = \frac{EA}{\frac{L}{5}} \begin{bmatrix} u_{2X} & u_{3X} \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} u_{2X} \\ u_{3X} \end{matrix}, \quad (2.45)$$

⑥ Add the column matrix of unknowns and external loads to complete the global system of equations:

$$\frac{EA}{\frac{L}{5}} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1+1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1+1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1+1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1+1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_{1X} \\ u_{2X} \\ u_{3X} \\ u_{4X} \\ u_{5X} \\ u_{6X} \end{bmatrix} = \begin{bmatrix} R_{1X} \\ 0 \\ R_{3X} \\ F_0 \\ 0 \\ R_{6X} \end{bmatrix}. \quad (2.50)$$

⑦ and ⑧ Introduce the boundary conditions to obtain the reduced system of equations. Solve the reduced system of equations to obtain the unknown nodal deformations.

There is no displacement possible at either ends of the structure (i.e., $u_{1X} = 0$ at node 1 and $u_{6X} = 0$ at node 6). Thus, cancel the first and last rows and the first and last columns from the linear system to obtain:

$$\frac{5EA}{L} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{3X} \\ u_{4X} \\ u_{5X} \end{bmatrix} = \begin{bmatrix} 0 \\ R_{3X} \\ F_0 \\ 0 \end{bmatrix}. \quad (2.51)$$

The first possibility to consider the non-homogeneous Dirichlet boundary condition $u_{3X} = u_0$ is:

$$\frac{5EA}{L} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{L}{5EA} & 0 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{3X} \\ u_{4X} \\ u_{5X} \end{bmatrix} = \begin{bmatrix} 0 \\ u_0 \\ F_0 \\ 0 \end{bmatrix}. \quad (2.52)$$

The solution can be obtained based on the matrix approach $\mathbf{u} = \mathbf{K}^{-1} \mathbf{f}$:

$$\begin{bmatrix} u_{2X} \\ u_{3X} \\ u_{4X} \\ u_{5X} \end{bmatrix} = \begin{bmatrix} \frac{u_0}{2} \\ u_0 \\ \frac{10EAu_0+2F_0L}{15EA} \\ \frac{5EAu_0+F_0L}{15EA} \end{bmatrix}. \quad (2.53)$$

The second possibility to consider the non-homogeneous Dirichlet boundary condition $u_{3X} = u_0$ is to multiply the 2nd column of the stiffness matrix with the given value u_0 :

$$\frac{5EA}{L} \begin{bmatrix} 2 & -1 \times u_0 & 0 & 0 \\ -1 & 2 \times u_0 & -1 & 0 \\ 0 & -1 \times u_0 & 2 & -1 \\ 0 & 0 \times u_0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{3X} \\ u_{4X} \\ u_{5X} \end{bmatrix} = \begin{bmatrix} 0 \\ R_{3X} \\ F_0 \\ 0 \end{bmatrix}. \quad (2.54)$$

Bring the second column to the right-hand side of the system of equations:

$$\frac{5EA}{L} \begin{bmatrix} 2 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{3X} \\ u_{4X} \\ u_{5X} \end{bmatrix} = \begin{bmatrix} 0 \\ R_{3X} \\ F_0 \\ 0 \end{bmatrix} - \frac{5EA}{L} \begin{bmatrix} -u_0 \\ 2u_0 \\ -u_0 \\ 0 \end{bmatrix}. \quad (2.55)$$

Now, let us cancel the second row of the system:

$$\frac{5EA}{L} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{4X} \\ u_{5X} \end{bmatrix} = \begin{bmatrix} 0 \\ F_0 \\ 0 \end{bmatrix} - \frac{5EA}{L} \begin{bmatrix} -u_0 \\ -u_0 \\ 0 \end{bmatrix}. \quad (2.56)$$

The solution can be obtained based on the matrix approach $\mathbf{u} = \mathbf{K}^{-1} \mathbf{f}$:

$$\begin{bmatrix} u_{2X} \\ u_{4X} \\ u_{5X} \end{bmatrix} = \begin{bmatrix} \frac{u_0}{2} \\ \frac{10EAu_0 + 2F_0L}{15EA} \\ \frac{5EAu_0 + F_0L}{15EA} \end{bmatrix}. \quad (2.57)$$

The third possibility to consider the non-homogeneous Dirichlet boundary condition $u_{3X} = u_0$ is to introduce the prescribed u_0 in the column matrix of unknowns:

$$\frac{5EA}{L} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_0 \\ u_{4X} \\ u_{5X} \end{bmatrix} = \begin{bmatrix} 0 \\ R_{3X} \\ F_0 \\ 0 \end{bmatrix}. \quad (2.58)$$

The column matrix of the nodal displacements $\mathbf{u}_p$ contains now unknown quantities (u_{2X}, u_{4X}, u_{5X}) and the given nodal boundary condition (u_0). On the other hand, the right-hand side contains the unknown reaction force R_{3X} . Thus, the structure of the linear system of equations is unfavorable for the solution. To rearrange the system to the classical structure where all unknowns are collected on the left and given quantities on the right hand side, the following steps can be applied:

Let us first split the right-hand side in known and unknowns quantities:

$$\frac{5EA}{L} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_0 \\ u_{4X} \\ u_{5X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F_0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ R_{3X} \\ 0 \\ 0 \end{bmatrix}. \quad (2.59)$$

Let us now multiply the second column of the stiffness matrix with the given value u_0 :

$$\frac{5EA}{L} \begin{bmatrix} 2 & -1 \times u_0 & 0 & 0 \\ -1 & 2 \times u_0 & -1 & 0 \\ 0 & -1 \times u_0 & 2 & -1 \\ 0 & 0 \times u_0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_0 \\ u_{4X} \\ u_{5X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F_0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ R_{3X} \\ 0 \\ 0 \end{bmatrix}. \quad (2.60)$$

The final step is to bring the second column of the stiffness matrix to the right-hand side of the system (known values) and the known column matrix with R_{3X} into the stiffness matrix:

$$\frac{5EA}{L} \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & -\frac{L}{5EA} & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_{2X} \\ R_{3X} \\ u_{4X} \\ u_{5X} \end{bmatrix} = \begin{bmatrix} \frac{5EAu_0}{L} \\ -\frac{10EAu_0}{L} \\ F_0 + \frac{5EAu_0}{L} \\ 0 \end{bmatrix}. \quad (2.61)$$

Now we can obtain the solution via the classical matrix approach $\mathbf{u} = \mathbf{K}^{-1} \mathbf{f}$:

$$\begin{bmatrix} u_{2X} \\ R_{3X} \\ u_{4X} \\ u_{5X} \end{bmatrix} = \begin{bmatrix} \frac{u_0}{2} \\ \frac{25EAu_0 - 4F_0L}{6L} \\ \frac{10EAu_0 + 2F_0L}{15EA} \\ \frac{5EAu_0 + F_0L}{15EA} \end{bmatrix}. \quad (2.62)$$

⑨ Post-computation: determination of reaction forces, stresses and strains.

Take into account the non-reduced system of equations as given in step ⑥ under the consideration of the known nodal displacements. The first equation of this system reads:

$$\frac{5EA}{L}(u_{1X} - u_{2X}) = R_{1X} \Rightarrow R_{1X} = -\frac{5EAu_0}{2L}. \quad (2.63)$$

In a similar way, we obtain from the other equations:

$$R_{3X} = \frac{25EAu_0 - 4F_0L}{6L}, \quad R_{6X} = -\frac{5EAu_0 + F_0L}{3L}. \quad (2.64)$$

The equations for the elemental strains, stresses, and normal forces can be extracted from Table 2.4:

$$\varepsilon_I^e = \frac{1}{L}(u_{2X} - u_{1X}) = \frac{u_0}{2L}, \quad (2.65)$$

$$\varepsilon_{II}^e = \frac{1}{L}(u_0 - u_{2X}) = \frac{u_0}{2L}, \quad (2.66)$$

$$\varepsilon_{III}^e = \frac{1}{L}(u_{4X} - u_0) = -\frac{5EAu_0 - 2F_0L}{15EAL}, \quad (2.67)$$

$$\varepsilon_{IV}^e = \frac{1}{L}(u_{5X} - u_{4X}) = -\frac{5EAu_0 + F_0L}{15EAL}, \quad (2.68)$$

$$\varepsilon_V^e = \frac{1}{L}(u_{6X} - u_{5X}) = -\frac{5EAu_0 + F_0L}{15EAL}, \quad (2.69)$$

$$\sigma_I^e = \frac{E}{L}(u_{2X} - u_{1X}) = \frac{Eu_0}{2L}, \quad (2.70)$$

$$\sigma_{II}^e = \frac{E}{L}(u_0 - u_{2X}) = \frac{Eu_0}{2L}, \quad (2.71)$$

$$\sigma_{III}^e = \frac{E}{L}(u_{4X} - u_0) = -\frac{5EAu_0 - 2F_0L}{15AL}, \quad (2.72)$$

$$\sigma_{IV}^e = \frac{E}{L}(u_{5X} - u_{4X}) = -\frac{5EAu_0 + F_0L}{15AL}, \quad (2.73)$$

$$\sigma_V^e = \frac{E}{L}(u_{6X} - u_{5X}) = -\frac{5EAu_0 + F_0L}{15AL}, \quad (2.74)$$

$$N_I^e = \frac{EA}{L}(u_{2X} - u_{1X}) = \frac{EAu_0}{2L}, \quad (2.75)$$

$$N_{II}^e = \frac{EA}{L}(u_0 - u_{2X}) = \frac{EAu_0}{2L}, \quad (2.76)$$

$$N_{III}^e = \frac{EA}{L}(u_{4X} - u_0) = -\frac{5EAu_0 - 2F_0L}{15L}, \quad (2.77)$$

$$N_{IV}^e = \frac{EA}{L}(u_{5X} - u_{4X}) = -\frac{5EAu_0 + F_0L}{15L}, \quad (2.78)$$

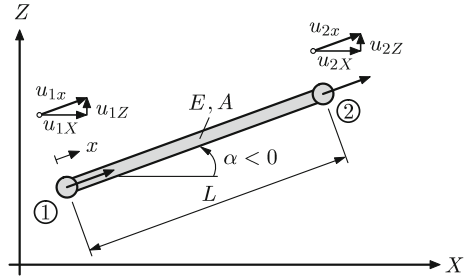
$$N_V^e = \frac{EA}{L}(u_{6X} - u_{5X}) = -\frac{5EAu_0 + F_0L}{15L}. \quad (2.79)$$

⑩ Check the global equilibrium between the external loads and the support reactions:

$$\sum_i F_{iX} = 0 \Leftrightarrow \underbrace{(R_{1X} + R_{3X} + R_{6X})}_{\text{reaction forces}} + \underbrace{F_0}_{\text{external load}} = 0. \quad \checkmark \quad (2.80)$$

Additional question: Assume now that only u_0 is given. Adjust the value of F_0 in such a way that element III is in a stress-free state. From ⑨ we can get the stress in element 3:

Fig. 2.14 Rotational transformation of a rod element in the X - Z plane



$$\sigma_{\text{III}}^e = -\frac{5AEu_0 - 2F_0L}{AL} \stackrel{!}{=} 0. \quad (2.81)$$

From the last equation we can conclude: $F_0 = \frac{5AEu_0}{2L}$.

2.3 Truss Structures

2.3.1 Revision of Theory

Let us consider a rod element which can deform in the global X - Z plane. The local x -coordinate is rotated by an angle α against the global coordinate system (X , Z), see Fig. 2.14. If the rotation of the global coordinate system to the local coordinate system is clockwise, a positive rotational angle is obtained.

Each node has now in the global coordinate system two degrees of freedom, i.e. a displacement in the X - and a displacement in the Z -direction. These two global displacements at each node can be used to calculate the displacement in the direction of the rod axis, i.e. in the direction of the local x -axis. The transformation of components of the principal finite element equation between the elemental and global coordinate system is summarized in Table 2.5 whereas the transformation matrix $\mathbf{T}$ is given by

$$\mathbf{T} = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 & 0 \\ 0 & 0 & \cos \alpha & -\sin \alpha \end{bmatrix}. \quad (2.82)$$

The triple matrix product for the stiffness matrix results in the following formulation for a rotated rod element:

Table 2.5 Transformation of matrices between the elemental (x, z) and global coordinate (X, Z) system

Stiffness matrix	
$\mathbf{K}_{xz}^e = \mathbf{T} \mathbf{K}_{XZ}^e \mathbf{T}^T$,	$\mathbf{K}_{XZ}^e = \mathbf{T}^T \mathbf{K}_{xz}^e \mathbf{T}$
Column matrix of nodal unknowns	
$\mathbf{u}_{xz} = \mathbf{T} \mathbf{u}_{XZ}$,	$\mathbf{u}_{XZ} = \mathbf{T}^T \mathbf{u}_{xz}$
Column matrix of external loads	
$\mathbf{f}_{xz} = \mathbf{T} \mathbf{f}_{XZ}$,	$\mathbf{f}_{XZ} = \mathbf{T}^T \mathbf{f}_{xz}$

$$\frac{EA}{L} \begin{bmatrix} \cos^2 \alpha & -\cos \alpha \sin \alpha & -\cos^2 \alpha & \cos \alpha \sin \alpha \\ -\cos \alpha \sin \alpha & \sin^2 \alpha & \cos \alpha \sin \alpha & -\sin^2 \alpha \\ -\cos^2 \alpha & \cos \alpha \sin \alpha & \cos^2 \alpha & -\cos \alpha \sin \alpha \\ \cos \alpha \sin \alpha & -\sin^2 \alpha & -\cos \alpha \sin \alpha & \sin^2 \alpha \end{bmatrix} \begin{bmatrix} u_{1X} \\ u_{1Z} \\ u_{2X} \\ u_{2Z} \end{bmatrix} = \begin{bmatrix} F_{1X} \\ F_{1Z} \\ F_{2X} \\ F_{2Z} \end{bmatrix}. \quad (2.83)$$

To simplify the solution of simple truss structures, Table 2.6 collects expressions for the global stiffness matrix for some common angles α .

The results for the transformation of matrices given in Table 2.5 can be combined with the relationships for post-processing of nodal values in Table 2.4 to express the distributions in global coordinates, see Table 2.7.

2.3.2 Worked Examples

2.4 Example: Simple Truss Structure with Two Members

Given is a plane truss structure as shown in Fig. 2.15. Both members have a uniform cross-sectional area A and Young's modulus E . The length of the members can be calculated from the given values (horizontal and vertical length a) in the figure. The structure is supported at its lower end and loaded by the single force F_0 .

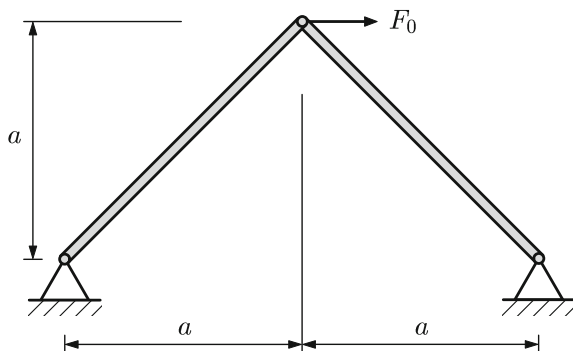
Model the truss structure with two linear finite elements and determine

- the displacement of the free node,
- the reaction forces at the supports,
- the strain, stress, and normal force in each element and
- check the global force equilibrium.

Table 2.6 Elemental stiffness matrices for truss elements given for different rotation angles α , cf. Eq. (2.83)

0°	180°
$\frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
-30°	30°
$\frac{EA}{L} \begin{bmatrix} \frac{3}{4} & \frac{1}{4}\sqrt{3} & -\frac{3}{4} & -\frac{1}{4}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & \frac{1}{4} & -\frac{1}{4}\sqrt{3} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{1}{4}\sqrt{3} & \frac{3}{4} & \frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & -\frac{1}{4} & \frac{1}{4}\sqrt{3} & \frac{1}{4} \end{bmatrix}$	$\frac{EA}{L} \begin{bmatrix} \frac{3}{4} & -\frac{1}{4}\sqrt{3} & -\frac{3}{4} & \frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{1}{4} & \frac{1}{4}\sqrt{3} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4}\sqrt{3} & \frac{3}{4} & -\frac{1}{4}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & -\frac{1}{4} & -\frac{1}{4}\sqrt{3} & \frac{1}{4} \end{bmatrix}$
-45°	45°
$\frac{EA}{L} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$	$\frac{EA}{L} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$
-90°	90°
$\frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$\frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

Fig. 2.15 Simple truss structure composed of two straight inclined members



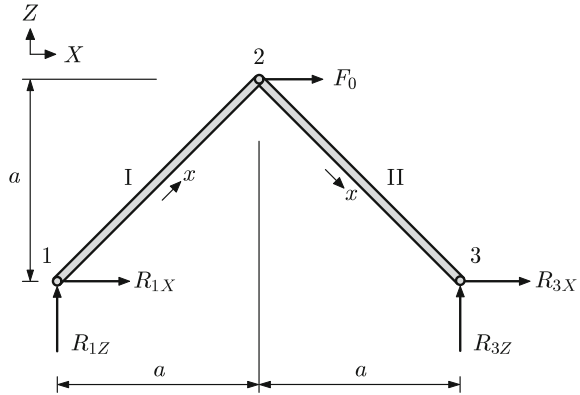
2.4 Solution

① and ② Sketch the free-body diagram of the problem, including a global coordinate system. Subdivide the geometry into finite elements. Indicate the node and

Table 2.7 Post-processing of nodal values in *global coordinates* for a linear rod element (defined by element length L , cross-sectional area A , and Young's modulus E)

	Axial displacement (elongation) u_x
$u_x^e(x) =$	$\left[1 - \frac{x}{L}\right] (\cos(\alpha)u_{1X} - \sin(\alpha)u_{1Z}) + \left[\frac{x}{L}\right] (\cos(\alpha)u_{2X} - \sin(\alpha)u_{2Z})$
$u_x^e(\xi) =$	$\left[\frac{1}{2}(1 - \xi)\right] (\cos(\alpha)u_{1X} - \sin(\alpha)u_{1Z}) + \left[\frac{1}{2}(1 + \xi)\right] (\cos(\alpha)u_{2X} - \sin(\alpha)u_{2Z})$
	Axial strain ε_x
$\varepsilon_x^e(x) =$	$\frac{1}{L} ((\cos(\alpha)u_{2X} - \sin(\alpha)u_{2Z}) - (\cos(\alpha)u_{1X} - \sin(\alpha)u_{1Z}))$
$\varepsilon_x^e(\xi) =$	$\frac{1}{L} ((\cos(\alpha)u_{2X} - \sin(\alpha)u_{2Z}) - (\cos(\alpha)u_{1X} - \sin(\alpha)u_{1Z}))$
	Axial stress σ_x
$\sigma_x^e(x) =$	$\frac{E}{L} ((\cos(\alpha)u_{2X} - \sin(\alpha)u_{2Z}) - (\cos(\alpha)u_{1X} - \sin(\alpha)u_{1Z}))$
$\sigma_x^e(\xi) =$	$\frac{E}{L} ((\cos(\alpha)u_{2X} - \sin(\alpha)u_{2Z}) - (\cos(\alpha)u_{1X} - \sin(\alpha)u_{1Z}))$
	Normal force N_x
$N_x^e(x) =$	$\frac{EA}{L} ((\cos(\alpha)u_{2X} - \sin(\alpha)u_{2Z}) - (\cos(\alpha)u_{1X} - \sin(\alpha)u_{1Z}))$
$N_x^e(\xi) =$	$\frac{EA}{L} ((\cos(\alpha)u_{2X} - \sin(\alpha)u_{2Z}) - (\cos(\alpha)u_{1X} - \sin(\alpha)u_{1Z}))$

Fig. 2.16 Free-body diagram of the truss structure composed of two straight inclined members



element numbers, local coordinate systems, and equivalent nodal loads, see Fig. 2.16.

③ Write separately all elemental stiffness matrices expressed in the global coordinate system. Indicate the nodal unknowns on the right-hand sides and over the matrices.

Element I is rotated by an angle of $\alpha = -45^\circ$:

$$\mathbf{K}_I^e = \frac{EA}{\sqrt{2}a} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_{1X} \\ u_{1Z} \\ u_{2X} \\ u_{2Z} \end{bmatrix}, \quad (2.84)$$

and element II is rotated by an angle of $\alpha = +45^\circ$:

$$\mathbf{K}_{II}^e = \frac{EA}{\sqrt{2}a} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{2Z} \\ u_{3X} \\ u_{3Z} \end{bmatrix}. \quad (2.85)$$

④ Determine the dimension of the global stiffness matrix and sketch the structure of this matrix with global unknowns on the right-hand side and over the matrix.

The finite element structure is composed of 3 nodes, each having two degree of freedom (i.e., the horizontal and vertical displacements). Thus, the dimensions of the global stiffness matrix are $(3 \times 2) \times (3 \times 2) = (6 \times 6)$:

$$K = \begin{bmatrix} u_{1X} & u_{1Z} & u_{2X} & u_{2Z} & u_{3X} & u_{3Z} \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \end{bmatrix} \begin{matrix} u_{1X} \\ u_{1Z} \\ u_{2X} \\ u_{2Z} \\ u_{3X} \\ u_{3Z} \end{matrix} . \quad (2.86)$$

⑤ Insert the values of the elemental stiffness matrices step-by-step into the global stiffness matrix:

$$K = \frac{EA}{\sqrt{2}a} \begin{bmatrix} u_{1X} & u_{1Z} & u_{2X} & u_{2Z} & u_{3X} & u_{3Z} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} + \frac{1}{2} & \frac{1}{2} - \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} - \frac{1}{2} & \frac{1}{2} + \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{matrix} u_{1X} \\ u_{1Z} \\ u_{2X} \\ u_{2Z} \\ u_{3X} \\ u_{3Z} \end{matrix} . \quad (2.87)$$

⑥ Add the column matrix of unknowns and external loads to complete the global system of equations:

$$\frac{EA}{\sqrt{2}a} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{2}{2} & 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{2}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_{1X} \\ u_{1Z} \\ u_{2X} \\ u_{2Z} \\ u_{3X} \\ u_{3Z} \end{bmatrix} = \begin{bmatrix} R_{1X} \\ R_{1Z} \\ F_0 \\ 0 \\ R_{3X} \\ R_{3Z} \end{bmatrix} . \quad (2.88)$$

⑦ Introduce the boundary conditions to obtain the reduced system of equations.

There is no displacement possible at the lower left-hand and lower right-hand of the structure (i.e., $u_{1X} = u_{1Z} = 0$ at node 1 and $u_{3X} = u_{3Z} = 0$ at node 3). Thus, cancel the first two and last two columns and rows from the linear system to obtain:

$$\frac{EA}{\sqrt{2}a} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{2Z} \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix} . \quad (2.89)$$

⑧ Solve the reduced system of equations to obtain the unknown nodal deformations.

The solution can be obtained based on the matrix approach $\mathbf{u} = \mathbf{K}^{-1} \mathbf{f}$:

$$\begin{bmatrix} u_{2X} \\ u_{2Z} \end{bmatrix} = \frac{\sqrt{2}a}{EA} \frac{1}{1-0} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix} = \frac{\sqrt{2}a F_0}{EA} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (2.90)$$

⑨ Post-computation: determination of reaction forces, stresses and strains.

Take into account the non-reduced system of equations as given in step ⑥ under the consideration of the known nodal displacements. The first equation of this system reads:

$$\frac{EA}{\sqrt{2}a} \left(-\frac{1}{2} u_{2X} \right) = R_{1X} \Rightarrow R_{1X} = -\frac{F}{2}. \quad (2.91)$$

In a similar way, we obtain from the other equations:

$$\frac{EA}{\sqrt{2}a} \left(-\frac{1}{2} u_{2X} \right) = R_{1Z} \Rightarrow R_{1Z} = -\frac{F}{2}, \quad (2.92)$$

$$\frac{EA}{\sqrt{2}a} \left(-\frac{1}{2} u_{2X} \right) = R_{3X} \Rightarrow R_{3X} = -\frac{F}{2}, \quad (2.93)$$

$$\frac{EA}{\sqrt{2}a} \left(\frac{1}{2} u_{2X} \right) = R_{3Z} \Rightarrow R_{3Z} = \frac{F}{2}. \quad (2.94)$$

The elemental stresses can be obtained from the displacements of the start ('s') and end ('e') node as:

$$\sigma = \frac{E}{\sqrt{2}a} (-\cos(\alpha)u_{sX} + \sin(\alpha)u_{sZ} + \cos(\alpha)u_{eX} - \sin(\alpha)u_{eZ}). \quad (2.95)$$

In the case of element I ($\alpha_I = -45^\circ$), we should consider $u_{1X} = u_{1Z} = u_{2Z} = 0$ to obtain:

$$\sigma_I = \frac{E}{\sqrt{2}a} \cos(\alpha_I) u_{2X} = \frac{F_0}{\sqrt{2}A}. \quad (2.96)$$

Similarly, $u_{1X} = u_{1Z} = u_{2Z} = 0$ for element II ($\alpha_{II} = +45^\circ$):

$$\sigma_{II} = -\frac{E}{\sqrt{2}a} \cos(\alpha_{II}) u_{2X} = -\frac{F_0}{\sqrt{2}A}. \quad (2.97)$$

Application of Hooke's law, i.e., $\sigma = E\varepsilon$, allows the calculation of the elemental strains:

$$\varepsilon_{\text{I}} = \frac{\sigma_{\text{I}}}{E} = +\frac{F_0}{\sqrt{2}EA}, \quad (2.98)$$

$$\varepsilon_{\text{II}} = \frac{\sigma_{\text{II}}}{E} = -\frac{F_0}{\sqrt{2}EA}. \quad (2.99)$$

The normal forces can be obtained from the normal stresses in each element:

$$N_{\text{I}} = \sigma_{\text{I}}A = \frac{F_0}{\sqrt{2}}, \quad (2.100)$$

$$N_{\text{II}} = \sigma_{\text{II}}A = -\frac{F_0}{\sqrt{2}}. \quad (2.101)$$

⑩ Check the global equilibrium between the external loads and the support reactions:

$$\sum_i F_{iX} = 0 \Leftrightarrow \underbrace{(R_{1X} + R_{3X})}_{\text{reaction force}} + \underbrace{F_0}_{\text{external loads}} = 0, \quad \checkmark \quad (2.102)$$

$$\sum_i F_{iZ} = 0 \Leftrightarrow \underbrace{(R_{1Z} + R_{3Z})}_{\text{reaction force}} + \underbrace{0}_{\text{external load}} = 0. \quad \checkmark \quad (2.103)$$

2.5 Example: Approximation of a Solid Using a Truss Structure

Given is an isotropic and homogeneous solid as shown in Fig. 2.17a. This solid should be modeled with the plane truss structure shown Fig. 2.17b. The six truss members have a uniform cross-sectional area A and Young's modulus E . The length of each member can be taken from the figure. The structure is supported at its left-hand side and the bottom. A uniform displacement u_0 is applied at the top nodes in the vertical direction.

Determine:

- the displacements of the nodes,
- the reaction forces at the supports and nodes where displacements are prescribed,
- the 'macroscopic' Poisson's ratio of the truss structure, and
- check the global force equilibrium.

2.5 Solution

The solution will follow the recommended 10 steps outlined on page 23.

① and ② Sketch the free-body diagram of the problem, including a global coordinate system. Subdivide the geometry into finite elements. Indicate the node and element numbers, local coordinate systems, and equivalent nodal loads, see Fig. 2.18.

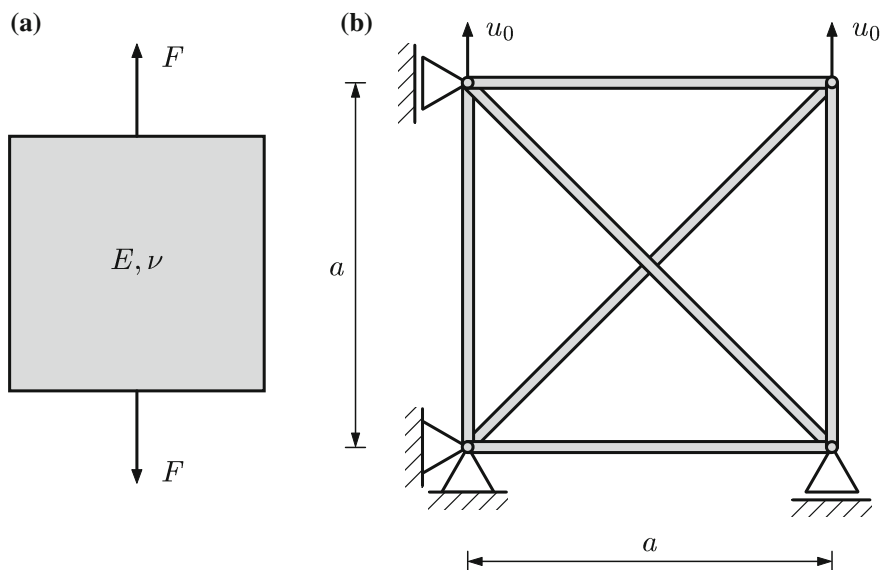
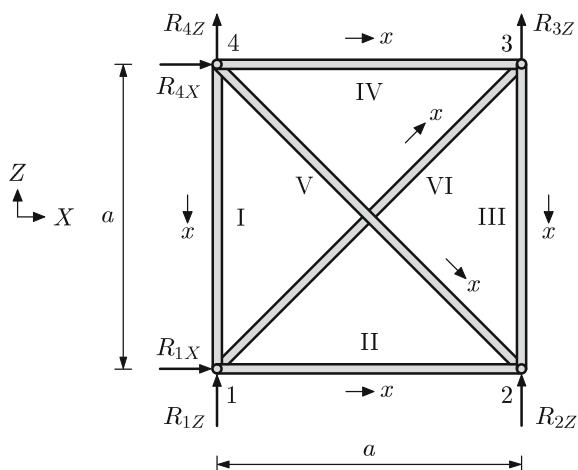


Fig. 2.17 Approximation of a solid using a truss: **a** solid, and **b** truss structure

Fig. 2.18 Free-body diagram of the truss structure



③ Write separately all elemental stiffness matrices expressed in the global coordinate system. Indicate the nodal unknowns on the right-hand sides and over the matrices.

Elements II and IV do not require any rotation ($\alpha = 0^\circ$) and thus, the simple elemental stiffness matrix given in Eq. (2.4) can be used:

$$\mathbf{K}_I^e = \frac{EA}{L} \begin{bmatrix} u_{1X} & u_{2X} \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_{1X} \\ u_{2X} \end{bmatrix}, \quad (2.104)$$

$$\mathbf{K}_{IV}^e = \frac{EA}{a} \begin{bmatrix} u_{4X} & u_{3X} \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_{4X} \\ u_{3X} \end{bmatrix}. \quad (2.105)$$

Elements I and III are rotated by an angle of $\alpha = +90^\circ$ and Eq. (2.83) allows to express the elemental stiffness matrices as:

$$\mathbf{K}_I^e = \frac{EA}{a} \begin{bmatrix} u_{4X} & u_{4Z} & u_{1X} & u_{1Z} \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{4X} \\ u_{4Z} \\ u_{1X} \\ u_{1Z} \end{bmatrix}, \quad (2.106)$$

$$\mathbf{K}_{III}^e = \frac{EA}{a} \begin{bmatrix} u_{3X} & u_{3Z} & u_{2X} & u_{2Z} \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{3X} \\ u_{3Z} \\ u_{2X} \\ u_{2Z} \end{bmatrix}. \quad (2.107)$$

Element V is rotated by an angle of $\alpha = +45^\circ$:

$$\mathbf{K}_V^e = \frac{EA}{\sqrt{2}a} \begin{bmatrix} u_{4X} & u_{4Z} & u_{2X} & u_{2Z} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_{4X} \\ u_{4Z} \\ u_{2X} \\ u_{2Z} \end{bmatrix}. \quad (2.108)$$

Element VI is rotated by an angle of $\alpha = -45^\circ$:

$$\mathbf{K}_{VI}^e = \frac{EA}{\sqrt{2}a} \begin{bmatrix} u_{1X} & u_{1Z} & u_{3X} & u_{3Z} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_{1X} \\ u_{1Z} \\ u_{3X} \\ u_{3Z} \end{bmatrix}. \quad (2.109)$$

④ Determine the dimensions of the global stiffness matrix and sketch the structure of this matrix with the global unknowns on the right-hand side and over the matrix.

The finite element structure is composed of 4 nodes, each having two degree of freedom (i.e., the horizontal and vertical displacements). Thus, the dimensions of the global stiffness matrix are $(4 \times 2) \times (4 \times 2) = (8 \times 8)$:

$$K = \begin{bmatrix} u_{1X} & u_{1Z} & u_{2X} & u_{2Z} & u_{3X} & u_{3Z} & u_{4X} & u_{4Z} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{matrix} u_{1X} \\ u_{1Z} \\ u_{2X} \\ u_{2Z} \\ u_{3X} \\ u_{3Z} \\ u_{4X} \\ u_{4Z} \end{matrix} . \quad (2.110)$$

⑤ Insert the values of the elemental stiffness matrices step-by-step into the global stiffness matrix.

$$\frac{K}{\frac{EA}{a}} = \begin{bmatrix} 1 + \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -1 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 \\ \frac{1}{2\sqrt{2}} & 1 + \frac{1}{2\sqrt{2}} & 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & -1 \\ -1 & 0 & 1 + \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & 1 + \frac{1}{2\sqrt{2}} & 0 & -1 & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 & 1 + \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -1 & 0 \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & -1 & \frac{1}{2\sqrt{2}} & 1 + \frac{1}{2\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -1 & 0 & 1 + \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & -1 & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 & -\frac{1}{2\sqrt{2}} & 1 + \frac{1}{2\sqrt{2}} \end{bmatrix} \begin{matrix} u_{1X} \\ u_{1Z} \\ u_{2X} \\ u_{2Z} \\ u_{3X} \\ u_{3Z} \\ u_{4X} \\ u_{4Z} \end{matrix} . \quad (2.111)$$

⑥ Add the column matrix of unknowns and external loads to complete the global system of equations.

The global system of equations can be expressed in matrix from as

$$K u_p = f , \quad (2.112)$$

where the column matrix of the external loads reads:

$$f = [R_{1X} \ R_{1Z} \ 0 \ R_{2Z} \ 0 \ R_{3Z} \ R_{4X} \ R_{4Z}]^T . \quad (2.113)$$

⑦ Introduce the boundary conditions to obtain the reduced system of equations.

The consideration of the support conditions, i.e. $u_{1X} = u_{1Z} = u_{2Z} = u_{4X} = 0$, results in the following 4×4 system:

$$\frac{EA}{a} \begin{bmatrix} 1 + \frac{1}{2\sqrt{2}} & 0 & 0 & \frac{1}{2\sqrt{2}} \\ 0 & 1 + \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{2\sqrt{2}} & 1 + \frac{1}{2\sqrt{2}} & 0 \\ \frac{1}{2\sqrt{2}} & 0 & 0 & 1 + \frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{3X} \\ u_{3Z} \\ u_{4Z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ R_{3Z} \\ R_{4Z} \end{bmatrix}. \quad (2.114)$$

The consideration of the displacement boundary condition $u_{3Z} = u_0$ allows a further reduction of the dimension of the system of equations. Multiplication of the third column of the coefficient matrix by the given displacement u_0 , bringing this column to the right-hand side of the system, and finally deletion of the third row results in the following equation:

$$\frac{EA}{a} \begin{bmatrix} 1 + \frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} \\ 0 & 1 + \frac{1}{2\sqrt{2}} & 0 \\ \frac{1}{2\sqrt{2}} & 0 & 1 + \frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{3X} \\ u_{4Z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ R_{4Z} \end{bmatrix} - \frac{EAu_0}{a} \begin{bmatrix} 0 \\ \frac{1}{2\sqrt{2}} \\ 0 \end{bmatrix}. \quad (2.115)$$

A further reduction can be achieved under the consideration of the displacement boundary conditions $u_{4Z} = u_0$. Multiplication of the third column of the coefficient matrix by the given displacement u_0 , bringing this column to the right-hand side of the system, and finally deletion of the third row results in:

$$\frac{EA}{a} \begin{bmatrix} 1 + \frac{1}{2\sqrt{2}} & 0 \\ 0 & 1 + \frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{3X} \end{bmatrix} = -\frac{EAu_0}{a} \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{bmatrix}. \quad (2.116)$$

⑧ Solve the reduced system of equations to obtain the unknown nodal deformations.

The solution can be obtained based on the matrix approach $\mathbf{u} = \mathbf{K}^{-1} \mathbf{f}$:

$$\begin{aligned} \begin{bmatrix} u_{2X} \\ u_{3X} \end{bmatrix} &= \frac{a}{EA} \times \frac{1}{\left(1 + \frac{1}{2\sqrt{2}}\right)^2 - 0} \begin{bmatrix} 1 + \frac{1}{2\sqrt{2}} & 0 \\ 0 & 1 + \frac{1}{2\sqrt{2}} \end{bmatrix} \left(-\frac{EAu_0}{a} \right) \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{bmatrix} \\ &= -\frac{u_0}{1 + 2\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned} \quad (2.117)$$

⑨ Post-computation: determination of reaction forces, stresses and strains.

Take into account the non-reduced system of equations as given in step ⑥ under the consideration of the known nodal displacements. The first equation of this system reads:

$$\frac{EA}{a} \left(-u_{2X} - \frac{u_{3X}}{2\sqrt{2}} - \frac{u_{3Z}}{2\sqrt{2}} + 0 \right) = R_{1X} \Rightarrow R_{1X} = 0. \quad (2.118)$$

The second equation of this system reads:

$$\frac{EA}{a} \left(0 - \frac{u_{3X}}{2\sqrt{2}} - \frac{u_{3Z}}{2\sqrt{2}} - u_{4Z} \right) = R_{1Z} \Rightarrow R_{1Z} = -\frac{4 + 2\sqrt{2}}{4 + \sqrt{2}} \times \frac{EAu_0}{a}. \quad (2.119)$$

In a similar way,³ the other reactions are obtained as:

$$R_{2Z} = R_{1Z}, R_{3Z} = R_{4Z} = -R_{1Z}, R_{4X} = 0. \quad (2.120)$$

The ‘macroscopic’ Poisson’s ratio of the truss structure is:

$$\nu = -\frac{\varepsilon_X}{\varepsilon_Z} = -\frac{-\frac{u_0/(1+2\sqrt{2})}{a}}{\frac{u_0}{a}} = \frac{1}{1 + 2\sqrt{2}} \approx 0.261. \quad (2.121)$$

⑩ Check the global equilibrium between the external loads and the support reactions:

$$\sum_i F_{iX} = 0 \Leftrightarrow \underbrace{(R_{1X} + R_{4X})}_{\text{reaction force}} + \underbrace{0}_{\text{external loads}} = 0, \quad \checkmark \quad (2.122)$$

$$\sum_i F_{iZ} = 0 \Leftrightarrow \underbrace{(R_{1Z} + R_{2Z} + R_{3Z} + R_{4Z})}_{\text{reaction force}} + \underbrace{0}_{\text{external load}} = 0. \quad \checkmark \quad (2.123)$$

2.6 Example: Truss Structure with Six Members (Computational Problem)

Given is a plane truss structure as shown in Fig. 2.19. The members have a uniform cross-sectional area A and Young’s modulus E . The length of each member can be taken from the figure. The structure is fixed at its left-hand side and loaded by

³The relation $\frac{4+2\sqrt{2}}{4+\sqrt{2}} = \frac{2(1+\sqrt{2})}{1+2\sqrt{2}}$ might be useful to show the identities.

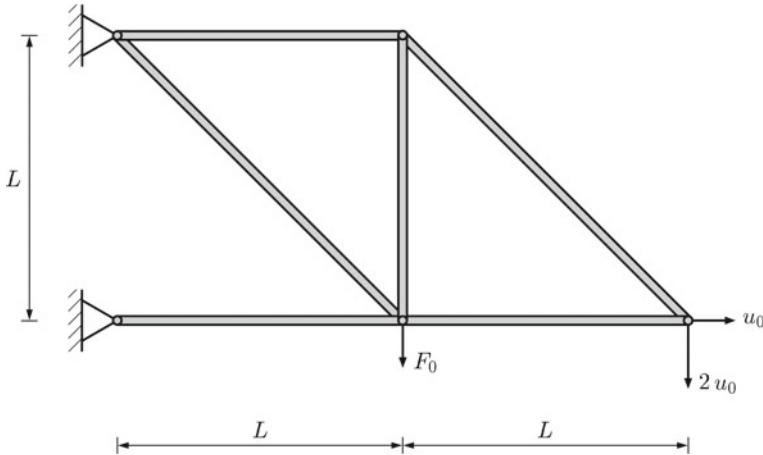


Fig. 2.19 Truss structure composed of six straight members

- two prescribed displacements u_0 and $2u_0$ at the very right-hand corner, and
- a vertical point load F_0 .

Model the truss structure with six linear finite elements and determine

- the displacements of the nodes,
- the reaction forces at the supports and nodes where displacements are prescribed,
- the strain, stress, and normal force in each element, and
- check the global force equilibrium.

Simplify all your results for the following special cases:

- $u_0 = 0$,
- $F_0 = 0$.

2.6 Solution

The solution will follow the recommended 10 steps outlined on page 23.

① and ② Sketch the free-body diagram of the problem, including a global coordinate system. Subdivide the geometry into finite elements. Indicate the node and element numbers, local coordinate systems, and equivalent nodal loads, see Fig. 2.20.

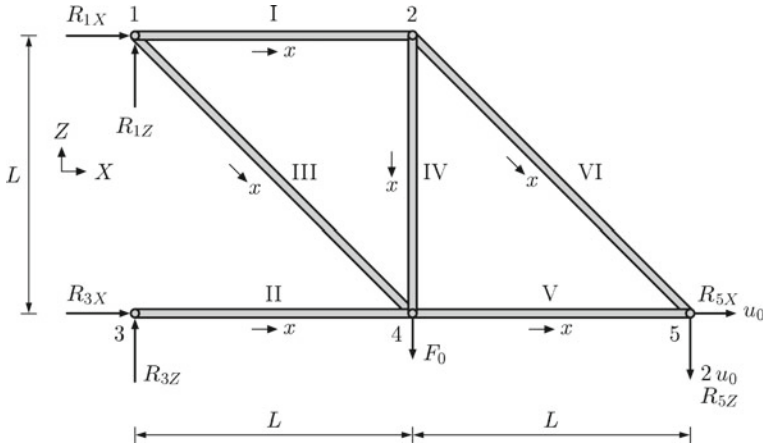


Fig. 2.20 Free-body diagram of the truss structure composed of six axial members

③ Write separately all elemental stiffness matrices expressed in the global coordinate system. Indicate the nodal unknowns on the right-hand sides and over the matrices.

Elements I, II and V do not require any rotation ($\alpha = 0^\circ$) and the simple elemental stiffness matrix given in Eq. (2.4) can be used:

$$\mathbf{K}_I^e = \frac{EA}{L} \begin{bmatrix} u_{1X} & u_{2X} \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} u_{1X} \\ u_{2X} \end{matrix}, \quad (2.124)$$

$$\mathbf{K}_{II}^e = \frac{EA}{L} \begin{bmatrix} u_{3X} & u_{4X} \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} u_{3X} \\ u_{4X} \end{matrix}, \quad (2.125)$$

$$\mathbf{K}_V^e = \frac{EA}{L} \begin{bmatrix} u_{4X} & u_{5X} \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} u_{4X} \\ u_{5X} \end{matrix}. \quad (2.126)$$

Element IV is rotated by an angle of $\alpha = +90^\circ$ and Eq. (2.83) allows to express the elemental stiffness matrix as:

$$\mathbf{K}_{IV}^e = \frac{EA}{L} \begin{bmatrix} u_{2X} & u_{2Z} & u_{4X} & u_{4Z} \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{matrix} u_{2X} \\ u_{2Z} \\ u_{4X} \\ u_{4Z} \end{matrix}. \quad (2.127)$$

Elements III and VI are both rotated by an angle of $\alpha = +45^\circ$:

$$K_{III}^e = \frac{EA}{\sqrt{2}L} \begin{bmatrix} u_{1X} & u_{1Z} & u_{4X} & u_{4Z} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_{1X} \\ u_{1Z} \\ u_{4X} \\ u_{4Z} \end{bmatrix} \quad (2.128)$$

$$= \frac{EA}{L} \begin{bmatrix} u_{1X} & u_{1Z} & u_{4X} & u_{4Z} \\ \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix} \begin{bmatrix} u_{1X} \\ u_{1Z} \\ u_{4X} \\ u_{4Z} \end{bmatrix}, \quad (2.129)$$

$$K_{VI}^e = \frac{EA}{\sqrt{2}L} \begin{bmatrix} u_{2X} & u_{2Z} & u_{5X} & u_{5Z} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{2Z} \\ u_{5X} \\ u_{5Z} \end{bmatrix} \quad (2.130)$$

$$= \frac{EA}{L} \begin{bmatrix} u_{2X} & u_{2Z} & u_{5X} & u_{5Z} \\ \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{2Z} \\ u_{5X} \\ u_{5Z} \end{bmatrix}. \quad (2.131)$$

④ Determine the dimensions of the global stiffness matrix and sketch the structure of this matrix with global unknowns on the right-hand side and over the matrix.

The finite element structure is composed of 5 nodes, each having two degrees of freedom (i.e., the horizontal and vertical displacements). Thus, the dimensions of the global stiffness matrix are $(5 \times 2) \times (5 \times 2) = (10 \times 10)$:

$$\mathbf{K} = \begin{bmatrix}
 u_{1X} & u_{1Z} & u_{2X} & u_{2Z} & u_{3X} & u_{3Z} & u_{4X} & u_{4Z} & u_{5X} & u_{5Z} \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
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 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
 \end{bmatrix} \begin{matrix} u_{1X} \\ u_{1Z} \\ u_{2X} \\ u_{2Z} \\ u_{3X} \\ u_{3Z} \\ u_{4X} \\ u_{4Z} \\ u_{5X} \\ u_{5Z} \end{matrix} \quad (2.132)$$

⑤ Insert the values of the elemental stiffness matrices step-by-step into the global stiffness matrix:

$$\frac{\mathbf{K}}{\frac{EA}{L}} = \begin{bmatrix}
 u_{1X} & u_{1Z} & u_{2X} & u_{2Z} & u_{3X} & u_{3Z} & u_{4X} & u_{4Z} & u_{5X} & u_{5Z} \\
 1 + \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -1 & 0 & 0 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 & 0 \\
 -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 \\
 -1 & 0 & 1 + \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\
 0 & 0 & -\frac{\sqrt{2}}{4} & 1 + \frac{\sqrt{2}}{4} & 0 & 0 & 0 & -1 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 & 0 & -1 & 0 & 1 + \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -1 & 0 \\
 \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & -1 & 0 & 0 & -\frac{\sqrt{2}}{4} & 1 + \frac{\sqrt{2}}{4} & 0 & 0 \\
 0 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 & 0 & -1 & 0 & 1 + \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
 0 & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}
 \end{bmatrix} \begin{matrix} u_{1X} \\ u_{1Z} \\ u_{2X} \\ u_{2Z} \\ u_{3X} \\ u_{3Z} \\ u_{4X} \\ u_{4Z} \\ u_{5X} \\ u_{5Z} \end{matrix} \quad (2.133)$$

⑥ and ⑦ Add the column matrix of unknowns and external loads to complete the global system of equations. Introduce the boundary conditions to obtain the reduced system of equations.

The global system of equations can be expressed in matrix from as

$$\mathbf{K} \mathbf{u}_p = \mathbf{f} \quad , \quad (2.134)$$

where the column matrix of the external loads reads:

$$\mathbf{f} = [R_{1X} \ R_{1Z} \ 0 \ 0 \ R_{3X} \ R_{3Z} \ 0 \ -F_0 \ R_{5X} \ -R_{5Z}]^T \quad . \quad (2.135)$$

The boundary conditions $u_{1X} = u_{1Z} = u_{3X} = u_{3Z} = 0$ allow to delete four rows and columns from the system of equations:

$$\frac{EA}{L} \begin{bmatrix} 1 + \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & 1 + \frac{\sqrt{2}}{4} & 0 & -1 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & 0 & 1 + 1 + \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -1 & 0 \\ 0 & -1 & -\frac{\sqrt{2}}{4} & 1 + \frac{\sqrt{2}}{4} & 0 & 0 \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -1 & 0 & 1 + \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{2Z} \\ u_{4X} \\ u_{4Z} \\ u_{5X} \\ u_{5Z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -F_0 \\ R_{5X} \\ -R_{5Z} \end{bmatrix}. \quad (2.136)$$

Let us first consider the displacement boundary condition $u_{5Z} = -2u_0$. Multiplication of the column corresponding to u_{5Z} with the prescribed value $-2u_0$ gives:

$$\frac{EA}{L} \begin{bmatrix} 1 + \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}(-2u_0) \\ -\frac{\sqrt{2}}{4} & 1 + \frac{\sqrt{2}}{4} & 0 & -1 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4}(-2u_0) \\ 0 & 0 & 2 + \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -1 & 0(-2u_0) \\ 0 & -1 & -\frac{\sqrt{2}}{4} & 1 + \frac{\sqrt{2}}{4} & 0 & 0(-2u_0) \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -1 & 0 & 1 + \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4}(-2u_0) \\ \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}(-2u_0) \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{2Z} \\ u_{4X} \\ u_{4Z} \\ u_{5X} \\ u_{5Z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -F_0 \\ R_{5X} \\ -R_{5Z} \end{bmatrix}. \quad (2.137)$$

Let us bring the last column of the stiffness matrix to the right-hand side and cancel the last row of the system of equations:

$$\frac{EA}{L} \begin{bmatrix} 1 + \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & 1 + \frac{\sqrt{2}}{4} & 0 & -1 & \frac{\sqrt{2}}{4} \\ 0 & 0 & 2 + \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -1 \\ 0 & -1 & -\frac{\sqrt{2}}{4} & 1 + \frac{\sqrt{2}}{4} & 0 \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -1 & 0 & 1 + \frac{\sqrt{2}}{4} \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{2Z} \\ u_{4X} \\ u_{4Z} \\ u_{5X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -F_0 \\ R_{5X} \end{bmatrix} - \frac{EA}{L} \begin{bmatrix} -\frac{\sqrt{2}}{2}u_0 \\ \frac{\sqrt{2}}{2}u_0 \\ 0 \\ 0 \\ \frac{\sqrt{2}}{2}u_0 \end{bmatrix}. \quad (2.138)$$

Now, let us consider the second prescribed displacement boundary condition $u_{5X} = u_0$. Multiplication of the column corresponding to u_{5X} with the prescribed value u_0 gives:

$$\frac{EA}{L} \begin{bmatrix} 1 + \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4}u_0 \\ -\frac{\sqrt{2}}{4} & 1 + \frac{\sqrt{2}}{4} & 0 & -1 & \frac{\sqrt{2}}{4}u_0 \\ 0 & 0 & 2 + \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -1u_0 \\ 0 & -1 & -\frac{\sqrt{2}}{4} & 1 + \frac{\sqrt{2}}{4} & 0u_0 \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -1 & 0 & (1 + \frac{\sqrt{2}}{4})u_0 \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{2Z} \\ u_{4X} \\ u_{4Z} \\ u_{5X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -F_0 \\ R_{5X} \end{bmatrix} - \frac{EA}{L} \begin{bmatrix} -\frac{\sqrt{2}}{4}u_0 \\ \frac{\sqrt{2}}{4}u_0 \\ 0 \\ 0 \\ \frac{\sqrt{2}}{4}u_0 \end{bmatrix}. \quad (2.139)$$

Let us bring the last column of the stiffness matrix to the right-hand side and cancel the last row of the system of equations:

$$\frac{EA}{L} \begin{bmatrix} 1 + \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 \\ -\frac{\sqrt{2}}{4} & 1 + \frac{\sqrt{2}}{4} & 0 & -1 \\ 0 & 0 & 2 + \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & -1 & -\frac{\sqrt{2}}{4} & 1 + \frac{\sqrt{2}}{4} \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{2Z} \\ u_{4X} \\ u_{4Z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -F_0 \end{bmatrix} - \frac{EA}{L} \left(\begin{bmatrix} -\frac{\sqrt{2}}{4}u_0 \\ \frac{\sqrt{2}}{4}u_0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{\sqrt{2}}{4}u_0 \\ \frac{\sqrt{2}}{4}u_0 \\ -u_0 \\ 0 \end{bmatrix} \right), \quad (2.140)$$

$$\frac{EA}{L} \begin{bmatrix} 1 + \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 \\ -\frac{\sqrt{2}}{4} & 1 + \frac{\sqrt{2}}{4} & 0 & -1 \\ 0 & 0 & 2 + \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & -1 & -\frac{\sqrt{2}}{4} & 1 + \frac{\sqrt{2}}{4} \end{bmatrix} \begin{bmatrix} u_{2X} \\ u_{2Z} \\ u_{4X} \\ u_{4Z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -F_0 \end{bmatrix} + \frac{EAu_0}{L} \begin{bmatrix} +\frac{3\sqrt{2}}{4} \\ -\frac{3\sqrt{2}}{4} \\ 1 \\ 0 \end{bmatrix}. \quad (2.141)$$

⑧ Solve the reduced system of equations to obtain the unknown nodal deformations.

The solution can be obtained based on the matrix approach $\mathbf{u} = \mathbf{K}^{-1} \mathbf{f}$:

$$u_{2X} = 0.429 u_0 - 0.408 \frac{FL}{EA}, \quad (2.142)$$

$$u_{2Z} = -1.357 u_0 - 1.562 \frac{FL}{EA}, \quad (2.143)$$

$$u_{4X} = 0.285 u_0 - 0.296 \frac{FL}{EA}, \quad (2.144)$$

$$u_{4Z} = -0.928 u_0 - 1.970 \frac{FL}{EA}. \quad (2.145)$$

⑨ Post-computation: determination of reaction forces, stresses and strains.

Take into account the non-reduced system of equations as given in step ⑥ under the consideration of the known nodal displacements. The evaluation of the first, second, fifth, sixth, ninth and tenth equation of this system gives the following results, respectively:

$$R_{1X} = -0.184 F_0 - 0.858 \frac{E A u_0}{L}, \quad (2.146)$$

$$R_{1Z} = 0.592 F_0 + 0.429 \frac{E A u_0}{L}, \quad (2.147)$$

$$R_{3X} = 0.296 F_0 - 0.285 \frac{E A u_0}{L}, \quad (2.148)$$

$$R_{3Z} = 0, \quad (2.149)$$

$$R_{5X} = -0.112 F_0 + 1.144 \frac{E A u_0}{L}, \quad (2.150)$$

$$R_{5Z} = -0.408 F_0 + 0.429 \frac{E A u_0}{L}. \quad (2.151)$$

The elemental stresses can be obtained from the displacements of the start ('s') and end ('e') node as:

$$\sigma = \frac{E}{L} (-\cos(\alpha)u_{sX} + \sin(\alpha)u_{sZ} + \cos(\alpha)u_{eX} - \sin(\alpha)u_{eZ}). \quad (2.152)$$

Application of this general equation (pay attention to the length of element III and VI which is equal to $\sqrt{2}L$) to the six elements under the consideration of the given nodal displacements gives:

$$\sigma_I = -0.408 \frac{F_0}{A} + 0.429 \frac{E u_0}{L}, \quad (2.153)$$

$$\sigma_{II} = -0.296 \frac{F_0}{A} + 0.285 \frac{E u_0}{L}, \quad (2.154)$$

$$\sigma_{III} = 0.837 \frac{F_0}{A} + 0.607 \frac{E u_0}{L}, \quad (2.155)$$

$$\sigma_{IV} = 0.408 \frac{F_0}{A} - 0.429 \frac{E u_0}{L}, \quad (2.156)$$

$$\sigma_V = 0.296 \frac{F_0}{A} + 0.715 \frac{E u_0}{L}, \quad (2.157)$$

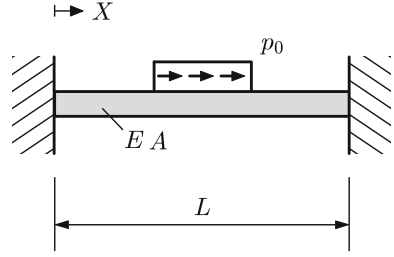
$$\sigma_{VI} = -0.577 \frac{F_0}{A} + 0.607 \frac{E u_0}{L}. \quad (2.158)$$

Application of Hooke's law, i.e., $\sigma = E\varepsilon$, allows the calculation of the elemental strains:

$$\varepsilon_I = -0.408 \frac{F_0}{EA} + 0.429 \frac{u_0}{L}, \quad (2.159)$$

$$\varepsilon_{II} = -0.296 \frac{F_0}{EA} + 0.285 \frac{u_0}{L}, \quad (2.160)$$

Fig. 2.21 Fixed rod structure with distributed load



$$\varepsilon_{\text{III}} = 0.837 \frac{F_0}{EA} + 0.607 \frac{u_0}{L}, \quad (2.161)$$

$$\varepsilon_{\text{IV}} = 0.408 \frac{F_0}{EA} - 0.429 \frac{u_0}{L}, \quad (2.162)$$

$$\varepsilon_{\text{V}} = 0.296 \frac{F_0}{EA} + 0.715 \frac{u_0}{L}, \quad (2.163)$$

$$\varepsilon_{\text{VI}} = -0.577 \frac{F_0}{EA} + 0.607 \frac{u_0}{L}. \quad (2.164)$$

⑩ Check the global equilibrium between the external loads and the support reactions:

$$\sum_i F_{iX} = 0 \Leftrightarrow \underbrace{(R_{1X} + R_{3X} + R_{5X})}_{\text{reaction force}} + \underbrace{0}_{\text{external loads}} = 0, \quad \checkmark \quad (2.165)$$

$$\sum_i F_{iZ} = 0 \Leftrightarrow \underbrace{(R_{1Z} + R_{3Z} - R_{5Z})}_{\text{reaction force}} + \underbrace{-F_0}_{\text{external load}} = 0. \quad \checkmark \quad (2.166)$$

2.4 Supplementary Problems

2.7 Fixed Rod Structure with Distributed Load

Given is a rod structure as shown in Fig. 2.21. The structure has a uniform cross-sectional area A and Young's modulus E . The structure is fixed at both ends and loaded by a distributed load p_0 in the range $\frac{L}{3} \leq X \leq \frac{2L}{3}$.

Model the rod structure with three linear finite elements of equal length and determine:

- the displacements at the nodes,
- the reaction forces at the supports,
- the strain, stress, and normal force in each element and
- check the global force equilibrium.

Fig. 2.22 Rod loaded under its dead weight

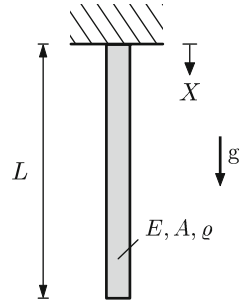
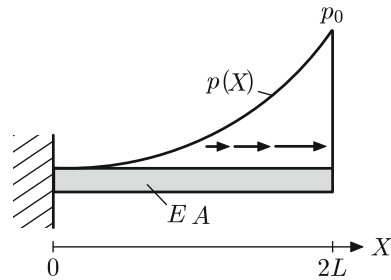


Fig. 2.23 Rod loaded under a distributed axial load



2.8 Rod Structure Under Dead Weight

Given is a rod structure which is deforming under the influence of its dead weight, see Fig. 2.22. The rod is of the original length L , cross-sectional area A , Young's modulus E , and mass density ρ . The standard gravity is given by g . Apply two linear rod elements of length $\frac{L}{2}$ to calculate the elongation of the rod due to its dead weight.

2.9 Rod Structure with Distributed Load

Given is a rod which is axially deforming under the influence of a distributed load $p(X) = \frac{p_0}{4} \left(\frac{X}{L}\right)^2$, see Fig. 2.23. The rod has the length L , cross-sectional area A , and Young's modulus E .

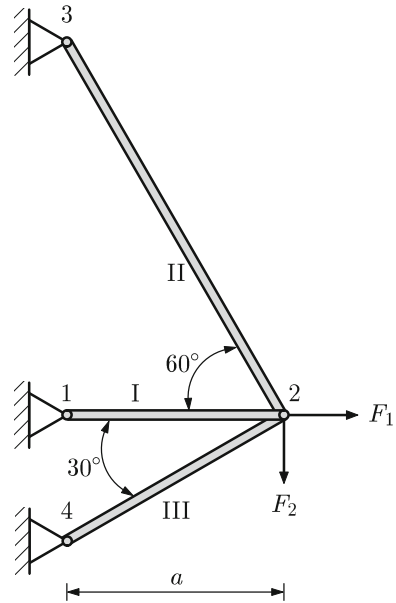
Apply two linear rod elements of length L to calculate the equivalent nodal loads for each element based on:

- analytical integration over the local x -coordinate,
- analytical integration over the natural coordinate $-1 \leq \xi \leq +1$,
- a one-point numerical integration rule and
- a two-point numerical integration rule.

2.10 Truss Structure with Three Members

Given is a plane truss structure as shown in Fig. 2.24. The members have a uniform cross-sectional area A and Young's modulus E . The length of each member can be taken from the figure. The structure is fixed at its left-hand sides and loaded by two points loads F_1 and F_2 .

Fig. 2.24 Truss structure composed of three straight members



Model the truss structure with three linear finite elements and determine:

- the displacements of the nodes,
- the reaction forces at the supports,
- the strain, stress, and normal force in each element and
- check the global force equilibrium.

2.11 Truss Structure in Star Formation

The following Fig. 2.25 shows a two-dimensional truss structure. The three rod elements have the same Young's modulus E and length L . However, the cross-sectional areas A_i ($i = \text{I, II, III}$) are different from the vertical rod (A_{I}) to those of the 45° inclined rods ($A_{\text{III}} = A_{\text{II}}$). The structure is loaded by a point load F_0 at node 2.

Develop a simplified (i.e., reduced number of rod elements) finite element truss structure under the consideration of the symmetry of the problem. Determine (do *not* consider three rod elements for the following questions):

- the equivalent statical system under the consideration of symmetry,
- the free-body diagram,
- the global stiffness matrix,
- the reduced system of equations under the consideration of the boundary conditions,
- the nodal displacement at node 2, and

Fig. 2.25 Three-element truss structure with force boundary condition

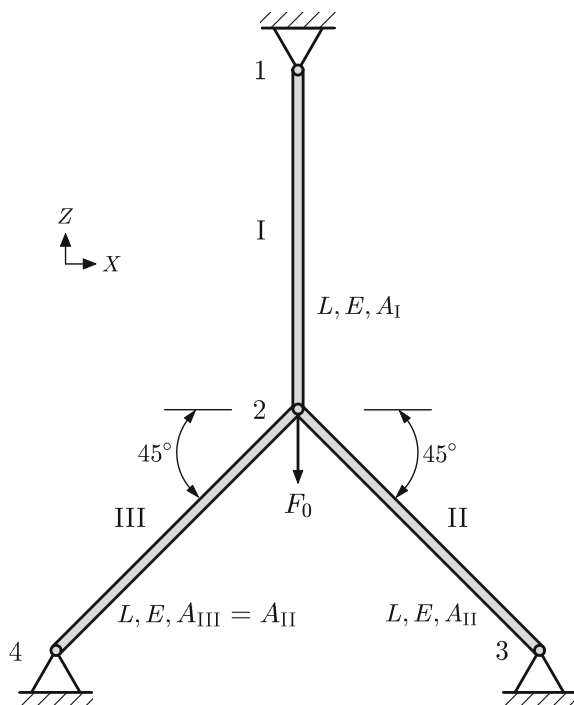
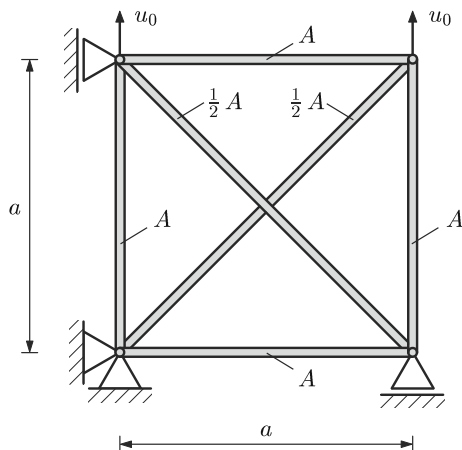


Fig. 2.26 Approximation of a solid due to a truss structure



- simplify the nodal displacement at node 2 for the special case $A_I = A_{II} = A_{III} = A$.

2.12 Approximation of a Solid—Based on a Truss Structure (Second Approach)

Given is an isotropic and homogeneous solid. This solid should be modeled with the plane truss structure shown in Fig. 2.26. The six truss members have a uniform

Young's modulus E . However, the cross-sectional areas are different, i.e., the diagonal members have only half of the area. The length of each member can be taken from the figure. The structure is supported at its left-hand side and the bottom. A uniform displacement u_0 is applied at the top nodes in the vertical direction.

Determine:

- the displacements of the nodes, and
- the 'macroscopic' Poisson's ratio of the truss structure.

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